九十四學年第一學期 PHYS2310 電磁學 期中考試題(共兩頁)

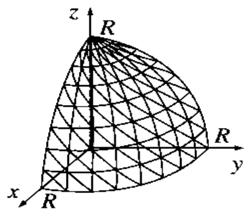
[Griffiths Ch. 1-3]　2005/11/15,　10:10am–12:00am, 教師：張存續

記得寫上學號，班別及姓名等。請依題號順序每頁答一題。

◇ Some useful formulas $\nabla\cdot\mathbf{v}=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta v_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}v_\phi$

1. (4%,4%,4%) Show that
   (a) $\delta(-x)=\delta(x)$,
   (b) $\delta(f(x))=\sum_i \frac{1}{\left|\frac{df}{dx}(x_i)\right|}\delta(x-x_i)$ where $f(x)$ is assumed to have only simple zero, located at $x=x_i$
   (c) $\delta(x^2-a^2)=\frac{1}{2|a|}[\delta(x-a)+\delta(x+a)]$ [Hint: use the result of (b)]

2. (4%, 6%) A vector function $\mathbf{v}=r^2\cos\theta\hat{\mathbf{r}}+r^2\cos\phi\hat{\boldsymbol{\theta}}-r^2\cos\theta\sin\phi\hat{\boldsymbol{\phi}}$.
   (a) Find $\nabla\cdot\mathbf{v}$ in spherical coordinate
   (b) Check the divergence theorem, $\oint\mathbf{v}\cdot d\mathbf{a}=\int_v(\nabla\cdot\mathbf{v})d\tau$, using as your volume one octant of the sphere of radius *R*. Write down all the surface integrals explicitly.



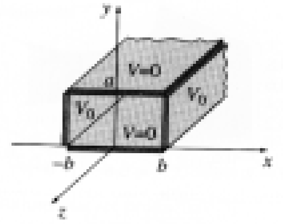
3. (4%, 4%, 4%, 4%) A metal sphere of radius *R*, carrying charge *q*, is surrounded by a thick concentric metal shell (inner radius *a*, outer radius *b*). The shell carries no net charge.
   (a) Find the surface charge density σ at *R*, at *a*, and at *b*.
   (b) Find the electric field at following two regions $R\le r\le a$ and $r\ge b$.
   (c) Find the potential at the center ($r=0$), using the infinity as the reference point.
   (d) If the outer shell is grounded, what is the potential of the inner sphere and what is the new surface charge density at *b*, $\sigma(b)$?

4. (4%, 4%, 4%)
   (a) Prove the normal component of **E** is discontinuous at any boundary, using Divergence theorem.
   (b) Prove the tangential component of **E** is always continuous, using Stoke's theorem.
   (c) Write down the normal and tangential component of electric fields immediately outside a metal surface with surface charge density σ.

5. (4%, 4%, 4%) The ratio of the magnitude of the charge on one conductor to the magnitude of the potential difference is called the capacitance, *C*. Find the capacitance of
   (a) two large, flat, conducting sheets of area *A*, separated by a small distance *d*;
   (b) two concentric conducting spheres with radii *a*, *b* ($b>a$);
   (c) two concentric conducting cylinders of length L, large compared to their radii *a*, *b* ($b>a$);

6. (7%, 7%) Separation of variable in spherical coordinate:
   (a) Two concentric conducting shells with radii *a*, *b* ($b>a$). The inner shell is connected to a potential of V_0, while the outer shell is grounded. Find the potential at $a<r<b$, and $r>b$.
   (b) The potential $V(R_0,\theta)=V_0\sin^2\theta$ is specified on the surface of a metal sphere, of radius R_0. Find the potential outside the sphere. [Hint: use Legentre polynomials] $P_0(x)=0,\ P_1(x)=x,$ and $P_2(x)=(3x^2-1)/2$

7. (4%, 5%, 5%) Two infinitely long grounded metal plates, again at y=0 and y=a, are connected at $x=\pm b$ by metal strips maintained at a constant potential V_0, as shown in the figure.
   (a) Write down the boundary conditions.
   (b) Write down the general solutions.
   (c) Find the potential inside the pipe.



8. (4%, 3%, 3%) An idea electric dipole **p** is situated at the origin, and points in the *z* direction. An electric charge *q*, of mass *m*, is released from rest at a point in the *xy* plane. The potential of the dipole is $V(\mathbf{r})=(1/4\pi\varepsilon_0)(p\cos\theta/r^2)$ and the gravitational force points in the –z direction.
   (a) Find the electric force between the dipole and the charge.
   (b) Find the total force (electrical and gravitational) on the charge.
   (c) Find the electrical potential energy.

1.

(a) $\int_{-\infty}^{\infty} f(x)\delta(-x)dx = -\int_{\infty}^{-\infty} f(-y)\delta(y)dy = \int_{-\infty}^{\infty} f(-y)\delta(y)dy = f(0)$

$\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0) \qquad \therefore \ \delta(-x) = \delta(x)$

(b)

$$\int_{-\infty}^{\infty} g(x)\delta(f(x))dx = \sum_i \int_{x_i-\varepsilon}^{x_i+\varepsilon} g(x)\delta(\left.\frac{df}{dx}\right|_{x_i}(x-x_i))dx$$

$$\because \int_{-\infty}^{\infty} g(x)\delta(k(x-x_i))dx = \frac{1}{|k|}g(x_i)$$

$$\Rightarrow \int_{x_i-\varepsilon}^{x_i+\varepsilon} g(x)\delta(\left.\frac{df}{dx}\right|_{x_i}(x-x_i))dx = \frac{1}{\left|\left.\frac{df}{dx}\right|_{x_i}\right|}g(x_i)$$

$$\therefore \int_{-\infty}^{\infty} g(x)\delta(f(x))dx = \sum_i \frac{1}{\left|\left.\frac{df}{dx}\right|_{x_i}\right|}g(x_i) \ \Rightarrow \ \delta(f(x)) = \sum_i \frac{1}{\left|\left.\frac{df}{dx}\right|_{x_i}\right|}\delta(x-x_i)$$

(c) Let $f(x) = x^2 - a^2$, $\delta(f(x)) = \sum_i \frac{1}{\left|\left.\frac{df}{dx}\right|_{x_i}\right|}\delta(x-x_i) = \frac{1}{|2a|}[\delta(x-a)+\delta(x+a)]$

2.

(a) $\mathbf{v} = r^2\cos\theta\hat{\mathbf{r}} + r^2\cos\phi\hat{\boldsymbol{\theta}} - r^2\cos\theta\sin\phi\hat{\boldsymbol{\varphi}}$

$$\nabla\cdot\mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2r^2\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta r^2\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(-r^2\cos\theta\sin\phi)$$

$$= 4r\cos\theta + r\frac{\cos\theta}{\sin\theta}\cos\phi - r\frac{\cos\theta}{\sin\theta}\cos\phi = 4r\cos\theta$$

(b) $\int_v (\nabla\cdot\mathbf{v})d\tau = \int_v 4r\cos\theta d\tau = \int_0^R\int_0^{\pi/2}\int_0^{\pi/2} 4r\cos\theta r^2\sin\theta drd\theta d\phi = R^4\frac{1}{2}\frac{\pi}{2} = \frac{\pi R^4}{4}$

$$\oint \mathbf{v}\cdot d\mathbf{a} = \int_{S_1} R^2\cos\theta R^2\sin\theta d\theta d\phi\left(=\frac{\pi R^4}{4}\right) + \int_{S_{xy}@\theta=\frac{\pi}{2}} (r^2\cos\phi)rdrd\phi\left(=\frac{1}{4}R^4\right)$$

$$+ \int_{S_{zx}@\phi=0} (-r^2\cos\theta\sin 0)rdrd\theta(=0) + \int_{S_{yz}@\phi=\frac{\pi}{2}} (-r^2\cos\theta\sin\frac{\pi}{2})rdrd\theta\left(=-\frac{1}{4}R^4\right)$$

$$= \frac{\pi R^4}{4}$$

3.

(a) $\sigma(R) = \frac{q}{4\pi R^2}$, $\sigma(a) = \frac{-q}{4\pi a^2}$, and $\sigma(b) = \frac{q}{4\pi b^2}$

(b) Use Gauss's law, we obtain $E = \frac{q}{4\pi\varepsilon_0 r^2}$ for both two regions, $R \le r \le a$ and $r \ge b$.

(c) $V(r) = -\int_{\infty}^{r} \mathbf{E} \cdot d\mathbf{r}$. $V(0) = V(R) = -\int_{\infty}^{b} \mathbf{E} \cdot d\mathbf{r} - \int_{a}^{R} \mathbf{E} \cdot d\mathbf{r} = \frac{q}{4\pi\varepsilon_0}(\frac{1}{b} + \frac{1}{R} - \frac{1}{a})$

(d) $V(r) = -\int_{a}^{r} \mathbf{E} \cdot d\mathbf{r}$, $V(R) = -\int_{a}^{R} \mathbf{E} \cdot d\mathbf{r} = \frac{q}{4\pi\varepsilon_0}(\frac{1}{R} - \frac{1}{a})$

$\sigma(b) = 0$

4.

(a) Consider a Gaussian pillbox. Gauss's law states that $\oint_S \mathbf{E} \cdot d\mathbf{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$

The sides of the pillbox contribute nothing to the flux, in the limit as the thickness ε goes to zero. $(E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp})A = \frac{\sigma A}{\varepsilon_0} \quad \Rightarrow \quad (E_{\text{above}}^{\perp} - E_{\text{below}}^{\perp}) = \frac{\sigma}{\varepsilon_0}$

(b) Consider a thin rectangular loop. The curl of the electric field states that $\oint_P \mathbf{E} \cdot d\ell = 0$

The ends gives nothing (as →ε0), and the sides give $(E_{\text{above}}^{//} - E_{\text{below}}^{//})\ell = 0 \quad \Rightarrow \quad E_{\text{above}}^{//} = E_{\text{below}}^{//}$

(c) Inside a metal, the electric field is zero $\mathbf{E}_{\text{below}} = 0$, so $\mathbf{E}_{\text{above}} = \frac{\sigma}{\varepsilon_0}\hat{\mathbf{n}} \quad \Rightarrow \quad \begin{cases} E_{above}^{\perp} = \frac{\sigma}{\varepsilon_0} \\ E_{above}^{//} = 0 \end{cases}$

5.

(a) $V = Ed = \frac{Q}{A\varepsilon_0}d$, $C = \frac{Q}{V} = \frac{\varepsilon_0 A}{d}$

(b) $V = -\int_{a}^{b} \mathbf{E} \cdot d\mathbf{r} = \frac{Q}{4\pi\varepsilon_0}(\frac{1}{a} - \frac{1}{b})$, $C = \frac{Q}{V} = 4\pi\varepsilon_0 \frac{ab}{b-a}$

(c) $V = -\int_{a}^{b} \mathbf{E} \cdot d\mathbf{r} = \frac{Q}{2\pi\varepsilon_0 L} \ln\frac{b}{a}$, $C = \frac{Q}{V} = 2\pi\varepsilon_0 L \Big/ \ln\frac{b}{a}$

6.

(a)

Boundary condition $\begin{cases} \text{(i) } V(a) = V_0 \\ \text{(ii) } V(b) = 0 \\ \text{(iii) } V(\infty) = 0 \end{cases}$

General solution $V(r,\theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos\theta) = A_0 + B_0 \frac{1}{r}$

$a < r < b$

$$\begin{cases} \text{B.C. (i) } \rightarrow A_0 + B_0 \frac{1}{a} = V_0 \\ \text{B.C. (ii) } \rightarrow A_0 + B_0 \frac{1}{b} = 0 \end{cases} \Rightarrow \begin{cases} B_0 = V_0 \frac{ab}{b-a} \\ A_0 = -V_0 \frac{a}{b-a} \end{cases} \quad \therefore V(r) = \frac{aV_0}{b-a}(-1 + \frac{b}{r})$$

$r > b$

$$\begin{cases} \text{B.C. (ii) } \rightarrow A_0 + B_0 \frac{1}{b} = 0 \\ \text{B.C. (iii) } \rightarrow A_0 = 0 \end{cases} \Rightarrow \begin{cases} A_0 = 0 \\ B_0 = 0 \end{cases} \quad \therefore V(r) = 0$$

(b)

Boundary condition $\begin{cases} \text{(i) } V(R_0,\theta) = V_0 \sin^2\theta \\ \text{(ii) } \lim_{r\to\infty} V(r,\theta) = 0 \end{cases}$

General solution $V(r,\theta) = \sum_{\ell=0}^{\infty} (A_\ell r^\ell + B_\ell r^{-(\ell+1)}) P_\ell(\cos\theta)$

B.C. (i) $\rightarrow A_\ell = 0$

B.C. (ii) $\rightarrow B_0 R_0^{-1} + B_1 R_0^{-2}\cos\theta + B_2 R_0^{-3}(\frac{3}{2}\cos^2\theta - \frac{1}{2}) = V_0(1-\cos^2\theta)$

$$\begin{cases} B_0 R_0^{-1} - \frac{1}{2} B_2 R_0^{-3} = V_0 \\ \frac{3}{2} B_2 R_0^{-3} = -V_0 \\ B_1 = 0 \end{cases} \Rightarrow \begin{cases} B_0 = \frac{2}{3} R_0 V_0 \\ B_1 = 0 \\ B_2 = -\frac{2}{3} R_0^3 V_0 \end{cases}$$

$\therefore V(r,\theta) = \frac{2R_0 V_0}{3r} - \frac{2R_0^3 V_0}{3r^3} P_2(\cos\theta)$

7.

(a) (i) $V = 0$ when $y = 0$,
(ii) $V = 0$ when $y = a$,
(iii) $V = V_0$ when $x = b$,
(iv) $V = V_0$ when $x = -b$.

(b) $V(x,y) = X(x)Y(y) \quad \Rightarrow \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = 0$

$$\begin{cases} \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = k^2 \quad \Rightarrow \quad X(x) = Ae^{kx} + Be^{-kx} \\ \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k^2 \quad \Rightarrow \quad Y(y) = C\sin ky + D\cos ky \end{cases}$$

$V(x,y) = (Ae^{kx} + Be^{-kx})(C\sin ky + D\cos ky)$

(c)

B.C.(i) $\Rightarrow \sin ka = 0,\quad k = \dfrac{n\pi}{a}\quad n = 1, 2, 3, \ldots$

B.C.(ii) $\Rightarrow D = 0$

B.C.(iii) + B.C.(iv) are symmetric, so $A = B \Rightarrow (Ae^{kx} + Be^{-kx}) = A\cosh(kx)$

$$V(x, y) = \sum_n C_n \cosh(kx)\sin ky$$

B.C.(iv) $V_0 = \sum_n C_n \cosh(\frac{n\pi}{a}b)\sin(\frac{n\pi}{a}y)$

$$C_n = \frac{2}{a}\frac{V_0}{\cosh(n\pi b/a)}\int_0^a \sin(\frac{n\pi}{a}y)dy = \frac{2}{a}\frac{V_0}{\cosh(\frac{n\pi}{a}b)}\frac{a}{n\pi}(2) = \frac{4}{n\pi}\frac{V_0}{\cosh(\frac{n\pi}{a}b)}$$

$$V(x, y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots}\frac{1}{n}\frac{\cosh(n\pi x/a)}{\cosh(n\pi b/a)}\sin(n\pi y/a)$$

8.

(a) $\mathbf{F}_e = q \cdot \mathbf{E} = -q\nabla V$ and $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{p\cos\theta}{r^2}$

$$\nabla V = \frac{\partial V}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\boldsymbol{\varphi}} = \frac{p}{4\pi\varepsilon_0}\left[-\frac{2\cos\theta}{r^3}\hat{\mathbf{r}} - \frac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}}\right]$$

$$\mathbf{F}_e = \frac{qp}{4\pi\varepsilon_0}\left[\frac{2\cos\theta}{r^3}\hat{\mathbf{r}} + \frac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}}\right]$$

(b) $\mathbf{F}_g = -mg\hat{\mathbf{z}} = mg(-\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}),\ \mathbf{F}_e = \dfrac{qp}{4\pi\varepsilon_0}\left[\dfrac{2\cos\theta}{r^3}\hat{\mathbf{r}} + \dfrac{\sin\theta}{r^3}\hat{\boldsymbol{\theta}}\right]$

$$\mathbf{F}_{total} = \mathbf{F}_g + \mathbf{F}_e = (-mg + \frac{2qp}{4\pi\varepsilon_0 r^3})\cos\theta\hat{\mathbf{r}} + (mg + \frac{qp}{4\pi\varepsilon_0 r^3})\sin\theta\hat{\boldsymbol{\theta}}$$

(c) $U_g = mgz = mgr\cos\theta,\ U_e = \dfrac{qp}{4\pi\varepsilon_0}\dfrac{\cos\theta}{r^2} \quad\Rightarrow\quad U_{total} = (mgr + \dfrac{qp}{4\pi\varepsilon_0 r^2})\cos\theta$